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Minimal nonsquare spectral factors

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Abstract

We consider the problem of parametrizing the set of all nonsquare minimal spectral factors of a rational matrix function taking positive semidefinite values on the imaginary axis. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

In the sequel, we will consider an $m \times m$ rational matrix function, $\Phi(\lambda)$, that has positive semidefinite values on the imaginary axis, $\mathrm{i}\mathbb{R}$. Note that, in this case, it is possible that Φ may have poles or zeros on $\mathrm{i}\mathbb{R}$. Furthermore, we shall mostly assume that $\Phi(\infty) = I_m$. The McMillan degree of Φ is always even as is well known (see, e.g., [20]) and is denoted by $2n$. We say that an $m \times p$ rational matrix function $W(\lambda)$ is a *minimal spectral factor* of $\Phi(\lambda)$ if

$$\Phi(\lambda) = W(\lambda)W(-\overline{\lambda})^* \tag{1}$$

is a minimal factorization. In other words, the McMillan degree of Φ is twice that of W. Here we denote the McMillan degree of W by $\delta(W)$. If Φ is regular, then W is

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regular in the sense that it has full row rank. We note that if $\Phi(\lambda) = W(\lambda)W(-\overline{\lambda})^*$, then Φ takes positive semidefinite values on the imaginary axis.

The spectral factorization problem in the multivariable case has an extensive history and originates from considerations in stochastic realization theory (see [6,23]), network synthesis (see [1]) and control (see [19]). In control the interest is mainly in one particular spectral factor, namely the one that is square, stable and has a stable inverse. In the stochastic realization problem one is interested in all stable spectral factors, and here also nonsquare stable spectral factors are of interest. In many papers the starting point is a decomposition of Φ, that is either additive as $\Phi(\lambda) = Z(\lambda) + Z(-\overline{\lambda})^*$ (as in [1,6,8,17]) or multiplicative as $\Phi(\lambda) = G(\lambda)G(-\overline{\lambda})^*$ (as in [7,10,11,16]). Formulas for the spectral factors are then given in terms of minimal realizations for $Z(\lambda)$ or $G(\lambda)$, respectively. As already mentioned, one is usually interested only in stable spectral factors associated with no poles or zeros on the imaginary axis. In [20] the starting point was a realization of Φ, where Φ is very general: poles and zeros are allowed on $\mathrm{i}\mathbb{R}$ and all (possibly nonstable) spectral factors are considered. Because it will not cause any extra difficulty we include also the nonstable spectral factors in our consideration.

In most papers, attention is focussed on square spectral factors (see [4,5,7,9,15,18,20,21]). In particular, we note that aspects of the problem of parametrizing the set of all square (i.e., $p = m$) minimal spectral factors of a rational matrix function taking positive semidefinite values on the imaginary axis were discussed in [18]. Here, the problem of parametrizing these spectral factors in terms of invariant subspaces and of the minimal unitary left divisors of a certain unitary function was considered. In this paper an approach that involves null-pole triples was adopted.

Nonsquare minimal spectral factors are discussed in several papers. In general, in the nonsquare situation, things are more complicated by comparison to the standard square case. From the background of stochastic realization theory it is discussed in [12–14]. In [10] the rectangular case is discussed from a geometric point of view involving shift invariant subspaces. Here all constraints on rank and zero location were removed with only very weak coercivity (Φ has constant rank, $p_0 \leqslant m$, on the extended imaginary axis) assumptions being made. The parametrization of the set $\mathscr{W}^p$ for a given p of all minimal stable, rectangular factors of size $m \times p$ for $p_0 \leqslant p \leqslant m + n$ was discussed. Moreover, in [11], state space formulas for matrix functions that arise from the analysis of singular, i.e., rectangular and not necessarily full rank, spectral factors, are derived. Here, formulas are established not only for the set of all stable spectral factors but also for all related inner functions (analytic on closed right halfplane, square and (co)isometric) originating from the analysis in [10]. (Recall that a matrix D is called an isometry if $D^*D = I$ and a coisometry if $DD^* = I$. A rational matrix function is said to take coisometric values if it does so on the imaginary line.)

The papers [16,17] treat the problem in a straightforward way; the starting point in [17] is an additive decomposition whereas the starting point in [16] is a multiplicative decomposition. In [17], a parametrization of all minimal stable nonsquare spectral factors is given in terms of an invariant subspace and the symmetric solution of an algebraic Riccati equation. In [16], spectral factorizations of descriptor continuous-time systems were computed by using the techniques of row compression by all-pass (square and (co)isometric) factors and dislocation of zeros by all-pass factors. The approaches adopted in [16,17] are closely connected to the methods used in [10,11] in the sense that the focus is on stable spectral factors and Φ has some regularity properties on $\mathrm{i}\mathbb{R}$.

In our discussion, we hope to show that similar results as those determined in [18,20] may be obtained in the case of nonsquare spectral factorizations of positive semidefinite rational matrix functions. Our starting point is different to those described previously; we start from a minimal realization of Φ (compare [20]). We shall assume throughout that Φ is regular, that is det $\Phi \not\equiv 0$. The most important element of our strategy is that it must take pure imaginary poles and zeros of Φ into account (as we did in [18,20], compare also [22]). The crucial idea is the following: let $\Phi(\lambda) = D + C(\lambda - A)^{-1}B$ be a minimal realization of Φ, and assume that Φ takes positive semidefinite values on the imaginary axis. Using $\Phi(\lambda) = \Phi(-\bar{\lambda})^*$ clearly $D = D^*$, and we see by Kalman's state space isomorphism theorem (see e.g., [2]) that there exists a unique invertible skew Hermitian matrix $H = -H^*$ such that $HA = -A^*H$ and $HBD = C^*$. (Mostly we shall take $D = I$ without loss of generality.) Then we use the fact that a minimum phase spectral factor W_1 shares a pole pair with a general minimal spectral factor W. This allows us to determine a realization for W. Although our result is stated completely in terms of the matrices appearing in a minimal realization of Φ, our proof makes use of the fact that we have a multiplicative decomposition of Φ. As a matter of fact, we rely heavily on the fact that we know the set of square spectral factors intimately from, e.g., [18,20].

Let us comment a little on our assumption that $\Phi(\infty) = I$. Obviously, since Φ is regular there is a $\lambda_0 \in \mathrm{i}\mathbb{R}$, not a pole of Φ, such that det $\Phi(\lambda_0) \neq 0$. Using a Möbius transform in λ, we can consider

$$\Psi(\lambda) = \Phi(\lambda)^{-1/2}\Phi\left(\frac{1}{\lambda - \lambda_0}\right) = \Phi(\lambda)^{-1/2}.$$

Clearly the minimal spectral factors of Φ and Ψ are related in a one-to-one manner, and $\Psi(\infty) = I$. This might even be worked out in state space formulas (see [2]), but for the sake of simplicity we shall refrain from doing that here.

The next section provides a parametrization originating from an invariant Lagrangian subspace and a certain Riccati inequality.

2. Minimal realizations for nonsquare minimal spectral factors

In the sequel, we will consider an $m \times m$ regular, rational matrix function, $\Phi(\lambda)$, that has positive semidefinite values on the imaginary axis, $\mathrm{i}\mathbb{R}$. We shall assume that $\Phi(\infty) = I_m$ and we denote $\delta(\Phi) = 2n$. Also, we shall allow that Φ takes complex

Hermitian matrices as its values on the imaginary line (the real case can be handled analogously).

The main problem that we wish to consider may be stated as follows. Given $\Phi \geqslant 0$ in realized form we wish to find all minimal spectral factorizations $\Phi = WW^*$, where W is possibly nonsquare. Our aim is to obtain a minimal realization for W's of this type. The approach that we will adopt in solving this problem is comparable to the one in [20].

In our first result, we investigate the connection between the pole pair of Φ and the pole pair of the minimal spectral factor W. Recall that a pair of matrices (C, A) is called a pole pair for Φ if there is a matrix B such that the realization $\Phi(\lambda) = I_m + C(\lambda I - A)^{-1}B$ is minimal.

Lemma 2.1. *Suppose that a minimal realization for $\Phi(\lambda)$ is given by*

$$\Phi(\lambda) = I_m + C(\lambda I - A)^{-1}B = I_m + C(\lambda I - A)^{-1}H^{-1}C^*, \tag{2}$$

where $H = -H^*$ and $HA = -A^*H$. If W is a minimal spectral factor, then W has a pole pair of the form $(C_1, A_1) = (C|_{\mathscr{M}}, A|_{\mathscr{M}})$, where $\mathscr{M}$ is an A-invariant, H-Lagrangian subspace.

Proof. We start by writing $W(\lambda) = D + C_1(\lambda I - A_1)^{-1}B_1$. Computing $\Phi(\lambda) = W(\lambda)W(-\overline{\lambda})^*$ from this realization for $W(\lambda)$ we see that $DD^* = I$ and that $\Phi(\lambda)$ has another minimal realization of the form

$$\Phi(\lambda) = I_m + \begin{pmatrix} C_1 & -DB_1^* \end{pmatrix}(\lambda I - \widetilde{A}_1)^{-1}) \begin{pmatrix} B_1D^* \\ C_1^*, \end{pmatrix}, \tag{3}$$

where

$$\widetilde{A}_1 = \begin{pmatrix} A_1 & -B_1B_1^* \\ 0 & -A_1^* \end{pmatrix}.$$

For the realization in (3) the corresponding H is given by

$$H_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Recall that H_1 is a unique skew-selfadjoint matrix for which

$$H_1\widetilde{A}_1 = -\widetilde{A}_1^*H_1 \quad \text{and} \quad H_1 \begin{pmatrix} B_1D^* \\ C_1^* \end{pmatrix} = \begin{pmatrix} C_1 & -DB_1^* \end{pmatrix}^*.$$

Because of minimality, realizations (2) and (3) are similar and as a result there exists a unique $\widetilde{S}$ such that

$$\widetilde{S}A = \widetilde{A}_1\widetilde{S}, \quad \widetilde{S}B = \begin{pmatrix} B_1D^* \\ C_1^* \end{pmatrix},$$
$$C\widetilde{S}^{-1} = \begin{pmatrix} C_1 & -DB_1^* \end{pmatrix} \quad \text{and} \quad H_1 = \widetilde{S}^{-*}H\widetilde{S}^{-1}.$$

We note that $\mathrm{im}\binom{I_n}{0}$ is H_1-Lagrangian and $\mathscr{M} = \widetilde{S}^{-1}\mathrm{im}\binom{I_n}{0}$ is an A-invariant, H-Lagrangian subspace, i.e., $\mathscr{M}$ is A-invariant and H-Lagrangian. This proves the lemma. $\square$

We are now in a position to state and prove our main result involving the parametrization of all nonsquare spectral factors.

Theorem 2.2. *Suppose that a positive semidefinite rational matrix function Φ has a minimal realization*

$$\Phi(\lambda) = I_m + C(\lambda I - A)^{-1}B. \tag{4}$$

There is a one-to-one correspondence between the set of minimal spectral factors $W(\lambda)$ of $\Phi(\lambda)$ such that $W(\infty) = (I_m \quad 0)$ and the set of triples $\{\mathscr{M}, X, \widehat{B}_1\}$. Here $\mathscr{M}$ is an A-invariant H-Lagrangian subspace. To describe X and $\widehat{B}_1$, let A_1 and C_1 be given by $A_1 = A|_{\mathscr{M}}$ and $C_1 = C|_{\mathscr{M}}$. Furthermore, suppose that $\mathscr{M}^\times$ is the $A^\times = (A - BC)$-invariant, H-Lagrangian subspace such that $\sigma(A^\times|_{\mathscr{M}^\times}) \subset \overline{\mathbb{C}}_-$. Let π be the projection onto $\mathscr{M}$ along $\mathscr{M}^\times$ and denote a matrix representation for πB by $\widetilde{B}_1$. Then in the triple $\{\mathscr{M}, X, \widehat{B}_1\}$ the matrix X solves the Riccati inequality

$$XC_1^*C_1X - X(A_1 - \widetilde{B}_1C_1)^* - (A_1 - \widetilde{B}_1C_1)X \leqslant 0 \tag{5}$$

and with such an X we take any $\widehat{B}_1$ such that

$$XC_1^*C_1X - X(A_1 - \widetilde{B}_1C_1)^* - (A_1 - \widetilde{B}_1C_1)X = -\widehat{B}_1\widehat{B}_1^*.$$

The correspondence between $W(\lambda)$ and the triple $\{\mathscr{M}, X, \widehat{B}_1\}$ is given by

$$W(\lambda) = (I_m \quad 0) + C_1(\lambda I - A_1)^{-1}(XC_1^* + \widetilde{B}_1 \quad \widehat{B}_1).$$

Proof. We use that without loss of generality, we can take $D = (I_m \quad 0)$. Indeed, as D is a coisometry, we can write it as $(I_m \quad 0)U$, where U is a unitary matrix. In this case, we can consider $W(\lambda)U^*$ instead of $W(\lambda)$. So, from here on, we assume that $W(\lambda)$ is normalized at infinity such that $W(\infty) = (I_m \quad 0)$. Also, we denote $B_1 = (B_{11} \quad B_{12})$.

First, we find the unique square minimal spectral factor $W_1(\lambda)$ corresponding to $\mathscr{M}$ and $\mathscr{M}^\times$. See [20], the main result there uses the factorization theorem of [2,3], see also [24]. This will be of the form

$$W_1(\lambda) = I_m + C_1(\lambda I - A_1)^{-1}\widetilde{B}_1 \tag{6}$$

for $\widetilde{B}_1$ as given in the theorem. From (6), we may compute Φ as

$$\Phi(\lambda) = W_1(\lambda)W_1(-\overline{\lambda})^* = I_m + (C_1 \quad -\widetilde{B}_1^*)(\lambda I - \widehat{A}_1)^{-1}\begin{pmatrix} \widetilde{B}_1 \\ C_1^* \end{pmatrix}, \tag{7}$$

where

$$\widehat{A}_1 = \begin{pmatrix} A_1 & -\widetilde{B}_1\widetilde{B}_1^* \\ 0 & -A_1^* \end{pmatrix}.$$

The H corresponding to realization (7) is the same as the H_1 for (3).

Let $W(\lambda)$ be a minimal spectral factor. Compute a realization for Φ from the realization $W(\lambda) = D + C_1(\lambda I - A_1)^{-1}B_1$ as in the proof of Lemma 3.1. This gives the realization (3) for Φ. Now, (3) and (7) are similar and the similarity S satisfies

$$S\widehat{A}_1 = \widetilde{A}_1 S, \quad S\begin{pmatrix} \widetilde{B}_1 \\ C_1^* \end{pmatrix} = \begin{pmatrix} B_{11} \\ C_1^* \end{pmatrix},$$
$$\begin{pmatrix} C_1 & -\widetilde{B}_1^* \end{pmatrix} S^{-1} = \begin{pmatrix} C_1 & -B_{11}^* \end{pmatrix} \quad \text{and} \quad S^* H_1 S = H_1.$$

From this, we note that S is H_1-unitary. (In other words, S is symplectic.)

From realizations (3) and (7), we compute the inverse of $\Phi(\lambda)$ in two ways:

$$\Phi(\lambda)^{-1} = I_m - \begin{pmatrix} C_1 & -B_1^* \end{pmatrix}(\lambda - \widetilde{A}_1^{\times})^{-1}\begin{pmatrix} B_1^* \\ C_1^* \end{pmatrix}$$

and

$$\Phi(\lambda)^{-1} = I_m - \begin{pmatrix} C_1 & -\widetilde{B}_1^* \end{pmatrix}(\lambda - \widehat{A}_1^{\times})^{-1}\begin{pmatrix} \widetilde{B}_1^* \\ C_1^* \end{pmatrix},$$

where

$$\widetilde{A}_1^{\times} = \begin{pmatrix} A_1 - B_{11}C_1 & -B_{12}B_{12}^* \\ -C_1C_1^* & -(A_1 - B_{11}C_1)^* \end{pmatrix}$$

and

$$\widehat{A}_1^{\times} = \begin{pmatrix} A_1 - \widetilde{B}_1C_1 & 0 \\ -C_1C_1^* & -(A_1 - \widetilde{B}_1C_1)^* \end{pmatrix},$$

respectively, and $S\widehat{A}_1^{\times} = \widetilde{A}_1^{\times} S$.

Put

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

If we consider the second block column $\binom{S_{12}}{S_{22}}$ of S we obtain

$$S\begin{pmatrix} 0 \\ -(A_1 - \widetilde{B}_1C_1)^* \end{pmatrix} = \begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}\left(-(A_1 - \widetilde{B}_1C_1)^*\right) = \widetilde{A}_1^{\times}\begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}. \tag{8}$$

In this case, it follows that the subspace $\mathscr{M}_0^{\times} = \operatorname{im}\binom{S_{12}}{S_{22}}$ is $\widetilde{A}_1^{\times}$-invariant, H_1-Lagrangian (from $S^*H_1S = H_1$) and $\widetilde{A}_1^{\times}|_{\mathscr{M}_0^{\times}}$ is similar to $-(A_1 - \widetilde{B}_1C_1)^*$. Since we have chosen W_1 such that W_1^{-1} has all its poles in the closed left halfplane (i.e., $\sigma(A_1 - \widetilde{B}_1C_1) \subset \overline{\mathbb{C}}_-$), then such an $\mathscr{M}_0^{\times}$ is unique. Now $\operatorname{im}\binom{I}{0}$ matches with any $\widetilde{A}_1^{\times}$-invariant, H_1-Lagrangian subspace (see [20]). (Recall that two subspaces $\mathscr{N}_1$ and $\mathscr{N}_2$ are said to *match* if $\mathscr{N}_1 \oplus \mathscr{N}_2 = \mathbb{C}^{2n}$.) In particular, $\operatorname{im}\binom{I}{0}$ must match with $\mathscr{M}_0^{\times}$ and as a result $\mathscr{M}_0^{\times} = \operatorname{im}\binom{X}{I}$, where $X = X^*$. Thus $S_{12} = XS_{22}$ and S_{22} is invertible.

We now have from (8) and $(C_1 \quad -B_{11}^*)S = (C_1 \quad -\widetilde{B}_1^*)$ that

$$-XS_{22}(A_1 - \widetilde{B}_1C_1)^* = (A_1 - B_{11}C_1)XS_{22} - B_{12}B_{12}^*S_{22}, \tag{9}$$

$$-S_{22}(A_1 - \widetilde{B}_1C_1)^* = -C_1^*C_1XS_{22} - (A_1 - B_{11}C_1)^*S_{22}, \tag{10}$$

$$(C_1X - B_{11}^*)S_{22} = -\widetilde{B}_1^*. \tag{11}$$

Now (10) implies that

$$(A_1 - B_{11}C_1)^* = -C_1^*C_1 + S_{22}(A_1 - \widetilde{B}_1C_1)^*S_{22}^{-1}.$$

Taking adjoints and inserting the result in (9) gives us

$$-XS_{22}(A_1 - \widetilde{B}_1C_1)^* = -XC_1^*C_1XS_{22} + S_{22}^{-*}(A_1 - \widetilde{B}_1C_1)S_{22}^*XS_{22}.$$

Multiplication by S_{22}^{-1} on the right gives us the algebraic Riccati equation

$$\begin{aligned} &XC_1^*C_1X - X(S_{22}(A_1 - \widetilde{B}_1C_1)^*S_{22}^{-1}) - (S_{22}^{-*}(A_1 - \widetilde{B}_1C_1)S_{22}^*)X \\ &\quad = -B_{12}B_{12}^*. \end{aligned} \tag{12}$$

Next, we prove that

$$S = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

For this, consider the minimal square spectral factor $W_1(\lambda)$. It has all its zeros in the closed left halfplane, and its pole pair is $(C_1 \quad A_1)$. This characterizes W_1 uniquely (see, e.g., [7,21]). Now, let $\hat{W}_1(\lambda)$ be the minimal square spectral factor that we obtain from the realization (3) for Φ and the supporting subspaces $\mathscr{M} = \operatorname{im}\binom{I}{0}$ and $\mathscr{M}_0^\times$, in the way described in [20]. That is, if we let Π be the projection along $\mathscr{M}$ onto $\mathscr{M}_0^\times$, then

$$\hat{W}_1(\lambda) = I_m + (C_1 \quad -B_{11}^*)|_{\mathscr{M}}(\lambda - \widetilde{A}_1|_{\mathscr{M}})^{-1}(I - \Pi)\begin{pmatrix} B_{11} \\ C_1^* \end{pmatrix}.$$

Now one easily sees that

$$\Pi = \begin{pmatrix} 0 & X \\ 0 & I \end{pmatrix}.$$

So

$$I - \Pi = \begin{pmatrix} I & -X \\ 0 & 0 \end{pmatrix}.$$

Thus

$$W_1(\lambda) = I_m + C_1(\lambda - A_1)^{-1}(B_{11} - XC_1^*).$$

Clearly, the zeros of $\hat{W}_1$ are the eigenvalues of $-(\widetilde{A}_1^\times|_{\mathscr{M}_0^\times})^*$ (compare Corollary 4.4 in [20]). So, the zeros of $\hat{W}_1$ are the eigenvalues of $A_1 - \widetilde{B}_1C_1$, that is, they coincide with the zeros of $W_1(\lambda)$. So we see that $\hat{W}_1$ has pole pair (C_1, A_1) and has all its zeros in the closed left halfplane. As already observed, this characterizes the minimal square spectral factor uniquely, and hence $\hat{W}_1(\lambda) = W_1(\lambda)$.

It follows that

$$C_1(\lambda - A_1)^{-1}(B_{11} - XC_1^*) = C_1(\lambda - A_1)^{-1}\widetilde{B}_1.$$

By observability of (C_1, A_1) we get that

$$B_{11} - XC_1^* = \widetilde{B}_1.$$

However, from (11) we have that

$$S_{22}^*(B_{11} - XC_1^*) = \widetilde{B}_1$$

and hence

$$(S_{22}^{-*} - I)\widetilde{B}_1 = 0. \tag{13}$$

Next, observe that $W_1(\lambda)$ is also obtained from the realization (7) by taking the subspaces $\mathscr{N} = \operatorname{im}\binom{I}{0}$ and $\mathscr{N}^\times = \operatorname{im}\binom{0}{I}$ as supporting subspaces. It then follows that the supporting projection corresponding to the factor $W_1(\lambda)$ for the realization (3) is given by the projection along $S\mathscr{N} = \mathscr{M}_0$ onto $S\mathscr{N}^\times = \mathscr{M}_0^\times$. Note that it follows that $\mathscr{M}_0 = \operatorname{im}\binom{S_{11}}{S_{21}}$. On the other hand, we have just seen that the supporting projection for the realization (3) corresponding to $W_1(\lambda)$ is the projection along $\mathscr{M} = \operatorname{im}\binom{I}{0}$ onto $\mathscr{M}_0^\times$. It follows that $\mathscr{M}_0 = \mathscr{M}$. So, S_{11} is invertible and $S_{21} = 0$.

Using the fact that S is H_1-unitary we see that $S_{11} = S_{22}^{-*}$.

From $S\hat{A}_1 = \widetilde{A}_1 S$ we obtain by looking at the $(1, 1)$-block entry that $S_{11}A_1 = A_1 S_{11}$, i.e.,

$$S_{22}^{-*}A_1 = A_1 S_{22}^{-*}. \tag{14}$$

So, $(S_{22}^{-*} - I)A_1 = A_1(S_{22}^{-*} - I)$. From this and (13) we see, using controllability of $(A_1, \widetilde{B}_1)$, that $S_{22} = I$.

Inserting this in (12) yields that X satisfies (5). We obtain also $B_{11} = XC_1^* + \widetilde{B}_1$. Writing $B_{12} = \widehat{B}_1$ we obtain

$$W(\lambda) = (I_m \quad 0) + C_1(\lambda I - A_1)^{-1}(XC_1^* + \widetilde{B}_1 \quad \widehat{B}_1).$$

Conversely, if $W(\lambda)$ is given by

$$W(\lambda) = (I_m \quad 0) + C_1(\lambda I - A_1)^{-1}(XC_1^* + \widetilde{B}_1 \quad \widehat{B}_1),$$

where X and $\widehat{B}_1$ are as above, then it is easy to see that W is a minimal spectral factor.

It remains to show that the correspondence between X and $W(\lambda)$ is one-to-one. It is obvious that different spectral factors give rise to different solutions X of the Riccati inequality. Conversely, assume that two solutions X_1 and X_2 of the Riccati inequality give rise to the same spectral factor $W(\lambda)$. Denote by $\hat{B}_{1i}$ the matrices for which both

$$X_i C_1^* C X_i - X_i(A_1 - \tilde{B}_1 C_1)^* - (A_1 - \tilde{B}_1 C_1)X_i = -\hat{B}_{1i}\hat{B}_{1i}^* \tag{15}$$

and

$$W(\lambda) = (I_m \quad 0) + C_1(\lambda I - A_1)^{-1}(X_i C_1^* + \widetilde{B}_1 \quad \widehat{B}_{1i})$$

hold. Using the observability of the pair (C_1, A_1) it follows that $B_{11} = B_{12}$, and that $X_1 C_1^* = X_2 C_1^*$. From Eq. (15) we then obtain that

$$X_1 A_1^* + A_1 X_1 = X_2 A_1^* + A_1 X_2,$$

that is,

$$A_1(X_1 - X_2) = (X_1 - X_2)A_1^*.$$

Hence $\mathrm{im}(X_1 - X_2) \subset \ker C_1$ and $A_1 \mathrm{im}(X_1 - X_2) \subset \mathrm{im}(X_1 - X_2)$. Again applying observability of (C_1, A_1) we see that $X_1 = X_2$. $\square$

Combining the Riccati equation from the theorem with the relation $B_{11} = XC_1^* + \widetilde{B}_1$ we can rewrite it as follows:

$$\begin{aligned} -B_{12}B_{12}^* &= XC_1^*C_1X + XC_1^*\widetilde{B}_1^* + \widetilde{B}_1C_1X + \widetilde{B}_1\widetilde{B}_1^* - XA_1^* - A_1X - \widetilde{B}_1\widetilde{B}_1^* \\ &= B_{11}B_{11}^* - XA_1^* - A_1X - \widetilde{B}_1\widetilde{B}_1^*. \end{aligned}$$

It follows that

$$XA_1^* + A_1X = B_{11}B_{11}^* + B_{12}B_{12}^* - \widetilde{B}_1\widetilde{B}_1^*,$$

that is,

$$XA_1^* + A_1X + \widetilde{B}_1\widetilde{B}_1^* = B_1B_1^*.$$

So we can rephrase the theorem as follows.

Theorem 2.3. *Suppose that a positive semidefinite rational matrix function Φ has a realization*

$$\Phi(\lambda) = I_m + C(\lambda I - A)^{-1}B. \tag{16}$$

There is a one-to-one correspondence between the set of minimal spectral factors $W(\lambda)$ of $\Phi(\lambda)$ such that $W(\infty) = (I_m \quad 0)$ and the set of triples $\{\mathscr{M}, X, B_1\}$. Here $\mathscr{M}$ is an A-invariant H-Lagrangian subspace. To describe X and B_1, let A_1 and C_1 be given by $A_1 = A|_{\mathscr{M}}$ and $C_1 = C|_{\mathscr{M}}$. Furthermore, suppose that $\mathscr{M}^\times$ is the $A^\times = (A - BC)$-invariant, H-Lagrangian subspace such that $\sigma(A^\times|_{\mathscr{M}^\times}) \subset \overline{\mathbb{C}}_-$. Let π be the projection onto $\mathscr{M}$ along $\mathscr{M}^\times$ and denote a matrix representation for πB by $\widetilde{B}_1$. Then X solves the Lyapunov inequality

$$XA_1^* + A_1X + \widetilde{B}_1\widetilde{B}_1^* \geqslant 0. \tag{17}$$

The correspondence is given by

$$W(\lambda) = (I_m \quad 0) + C_1(\lambda I - A_1)^{-1}B_1,$$

where B_1 satisfies

$$XA_1^* + A_1X + \widetilde{B}_1\widetilde{B}_1^* = B_1B_1^*. \tag{18}$$

It is interesting to note that in [11] the authors started with a realization for a stable minimal nonsquare spectral factor W. Subsequently they proceeded to construct W_1 that has the same pole pair as W but is square minimum phase. Then they analyse the inner factors $W_1^{-1}W$. By contrast to their approach, our method involved starting with a realization for Φ. Next, we deduced a result involving the pole pair of W_1 (see Lemma 3.1) and constructed by previous results (see [20]) a minimum phase W_1 with the same pole pair as W. From this we could determine a realization for W (see Theorem 3.2). An alternative approach to the one adopted by us may also be outlined as follows. Given a realization for W, we can construct a realization for W_1. In this regard, we observe from the proof of Theorem 3.2 that

$$\begin{aligned}
&XA_1^* + A_1X - XC_1^*C_1X - B_{12}B_{12}^* \\
&\quad = XC_1^*(B_{11} - XC_1^*)^* + (B_{11} - XC_1^*)C_1X \\
&\quad = -2XC_1^*C_1X + XC_1^*B_{11} + B_{11}C_1X.
\end{aligned}$$

Consequently, we have that

$$XC_1^*C_1X + XA_1^* + A_1X - B_{12}B_{12}^* - XC_1^*B_{11} - B_{11}C_1X = 0.$$

Note that $\widetilde{B}_1 = B_{11} - XC_1^*$, where X is such that $A_1 - \widetilde{B}_1C_1 = A_1 - B_{11}C_1 - XC_1^*C_1$ has all its eigenvalues in the closed left halfplane. In other words, we have shown that, as in [11], we can also construct W_1 from W.

Our results may be compared to the work done by Pavon in [17], where a parametrization of all minimal stable nonsquare spectral factors is given in terms of an invariant subspace and the symmetric solution of an algebraic Riccati equation. In order to explain the connection to our results, let $\Phi(\lambda) = I + C(\lambda - A)^{-1}B$ be a minimal realization of Φ and assume that $H = -H^*$ is the unique invertible skew Hermitian matrix such that $HA = -A^*H$ and $HB = -C^*$. Suppose that $\mathscr{M}$ is a subspace with the properties that $A\mathscr{M} \subset \mathscr{M}$ and $H\mathscr{M} = \mathscr{M}^{\perp}$. In order to establish a connection with [17], we assume that $\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$. Furthermore, we take $\mathscr{M}$ to be the stable invariant subspace (i.e., $\sigma(A|_{\mathscr{M}}) = \sigma(A_1) \subset \mathbb{C}_-$). Next, we consider

$$\widetilde{A}_1 = \begin{pmatrix} A_1 & -B_1B_1^* \\ 0 & -A_1^* \end{pmatrix}$$

in

$$\Phi(\lambda) = I_m + (C_1 \quad -DB_1^*)(\lambda I - \widetilde{A}_1)^{-1}\begin{pmatrix} B_1D^* \\ C_1^* \end{pmatrix},$$

(compare formula (3) in the proof of Lemma 3.1) and let Q solve the Lyapunov equation

$$A_1Q + QA_1^* = B_1B_1^*.$$

We see that for

$$T = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix},$$

we have

$$T^{-1}\widetilde{A}_1 T = \begin{pmatrix} A_1 & 0 \\ 0 & -A_1^* \end{pmatrix},$$

and hence

$$\begin{aligned} \Phi(\lambda) &= I_m + (C_1 \quad C_1Q - DB_1^*)\left(\lambda I - \begin{pmatrix} A_1 & 0 \\ 0 & -A_1^* \end{pmatrix}\right)^{-1}\begin{pmatrix} B_1D^* - QC_1^* \\ C_1^* \end{pmatrix} \\ &= I_m + C_1(\lambda I - A_1)^{-1}(B_1D^* - QC_1^*) + (C_1Q - DB_1^*)(\lambda I - A_1^*)^{-1}C_1^*. \end{aligned}$$

As was mentioned earlier, such an additive decomposition is the starting point in [17] (and also in many others papers, e.g., in [1,6]). An important conclusion is that the Riccati inequality given by (5) above reduces to the Riccati inequality appearing in [17] (see inequality (4) there).

We finish with several remarks concerning our main result.

Remark 1. The first remark involves the description of the minimal square spectral factors from Theorem 3.2. Obviously, $W(\lambda)$ is square if and only if $\hat{B}_1$ is zero on the zero dimensional space. This means that X solves the Riccati equation

$$XC_1^*C_1X - X(A_1 - \tilde{B}_1C_1) - (A_1 - \tilde{B}_1C_1)^*X = 0.$$

Moreover, minimal square spectral factors are of the form

$$W(\lambda) = I_m + C_1(\lambda I - A_1)^{-1}(XC_1^* + \tilde{B}_1).$$

The inverse is then given by

$$W(\lambda)^{-1} = I_m - C_1(\lambda I - (A_1 - \tilde{B}_1C_1 - XC_1^*C_1))^{-1}(XC_1^* + \tilde{B}_1).$$

Just taking $\hat{B}_1 = 0$ on a space of dimension larger than or equal to 1 gives an interesting class of minimal spectral factors: these are the ones with $W(\infty) = (I_m \quad 0)$ that are of the form $\tilde{W}(\lambda)(I_m \quad 0)$, where $\tilde{W}$ is a minimal square spectral factor with $\tilde{W}(\infty) = I_m$.

Remark 2. We also note that, with notation as in the proof of Theorem 3.2, we have that $V(\lambda) = W_1(\lambda)^{-1}W(\lambda)$ is a rational matrix function with coisometric values on $\mathrm{i}\mathbb{R}$. We compute $V(\lambda)$ explicitly from

$$\begin{aligned} V(\lambda) = {}& (I_m - C_1(\lambda I - (A_1 - \tilde{B}_1C_1))^{-1}\tilde{B}_1)((I_m \quad 0) \\ &+ C_1(\lambda I - A_1)^{-1}(XC_1^* + \tilde{B}_1 \quad \hat{B}_1)). \end{aligned}$$

Writing out the product and using that

$$\tilde{B}_1C_1 = (\lambda - (A_1 - \tilde{B}_1C_1)) - (\lambda - A_1),$$

we see that

$$V(\lambda) = (I_m \quad 0) + C_1(\lambda I - (A_1 - \tilde{B}_1 C_1))^{-1}(XC_1^* \quad \hat{B}_1)$$

(compare [11]).

Remark 3. In case $\Phi(\lambda)$ has no poles on the imaginary axis, we may obtain all stable spectral factors simply by taking for $\mathscr{M}$ the stable invariant subspace of A, i.e., the subspace spanned by eigenvectors and generalized eigenvectors of A corresponding to eigenvalues in the open left halfplane.

3. Conclusion

We have given two parametrizations for minimal possibly nonsquare spectral factors of a given positive semidefinite $\Phi(\lambda)$ (Theorems 2.2 and 2.3). The parametrizations provide minimal realizations for the factors. They require us to find an invariant Lagrangian subspace, then to solve an algebraic Riccati inequality, followed by a symmetric decomposition of a matrix. Our method allows for poles and zeros of $\Phi(\lambda)$ on the imaginary axis, and provides also all nonstable spectral factors.

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